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Transport Equation for a Fermi System in Random Scattering Centers. II. Independent Electrons in an Arbitrarily Varying Electric Field and Strong Single-Center Potentials*

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An application of a diagrammatic technique, given by us recently, is made for the calculation of the coefficients of a transport equation for dynamically independent electrons in random impurities. The coefficients of the transport equation are given for arbitrary wavelength and frequency of the electric field (transverse or longitudinal) and strong single-center potentials, but for low impurity concentration.

I. INTRODUCTION

The quantum theory of transport of the simplified but physical model of a real substance consisting of dynamically independent electrons interacting only with a random array of impurities has received considerable attention.[1–14] Various methods of deriving, without untenable assumptions, a transport equation for the distribution function from the Schrödinger equation have been employed.

The nature of the coefficients of this transport equation has been studied in some detail in various special cases. In the case of a *uniform* and *steady* electric field these coefficients have been given[1,3,6,12,14] in a power series in a parameter λ that is a dimensionless measure of the strength of the interaction of the electrons with *all* the impurities. In the lowest order in λ, the coefficients are given in terms of the concentration of the impurities and the Born approximation to the scattering cross section of one impurity. In higher orders in λ, higher approximations for the one-impurity scattering cross section come in, as well as the cross section for scattering off two impurities, etc. Luttinger and Kohn[2] have also given expressions for these coefficients in powers of the density of the impurities and in terms of the *total* cross section of one, two, etc., impurities, i.e., expressions valid for arbitrary strength of the interaction of the electron with each impurity, but for low impurity concentrations. Even for this simple case of uniform and steady electric field, the calculations[2] become, unfortunately quite involved and thus no clear way of treating more general cases of physical interest is evident.

In the case of an *inhomogeneous* and *time-varying* electric field, these coefficients turn out to depend on both the wave vector $\mathbf{q}$ and the frequency ω of the electric field and have been studied[8,9,13] only in the lowest order in λ, i.e., only in the Born approximation for the strength of the one-impurity potential and the first power of the impurity concentration. The traditional Boltzmann equation for the distribution function then results if these coefficients are expanded in powers of $\mathbf{q}$ and ω and, as in the case of smooth and slowly varying electric fields, only the leading terms are kept. Beyond these approximations the Boltzmann equation is *not* valid.

In this paper we consider the case of an electric field of arbitrary $\mathbf{q}$ and ω and derive the coefficients of the quantum transport equation in powers of the impurity density and for arbitrary strength of the one-impurity potential. Thus, the results of Luttinger and Kohn[2] are special cases of these more general expressions. Although no applications will be made here, this more general case of finite $\mathbf{q}$ and ω is of obvious physical interest in the field of semiconductors and metals.

The method we shall follow is the one expanded recently by us.[15] Although this method was primarily developed for a system of interacting fermions, it proves very convenient and relatively easy for this problem as well.

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The Hamiltonian of the system of free electrons in the field of a set of impurities,

$$H=H_0+V_i, \tag{1.1}$$

consists of the kinetic energy

$$H_0=\sum_k \epsilon_k c_k{}^\dagger c_k, \quad \epsilon_k=\mathbf{k}^2/2m, \tag{1.2}$$

where k denotes the wave vector $\mathbf{k}$ of a plane-wave state and its spin σ, and the interaction energy of the particles with the impurities at positions $\mathbf{R}_j$

$$V_i=\sum_{kk'} [u(k-k')\sum_j e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}_j}]c_k{}^\dagger c_{k'}. \tag{1.3}$$

Here, $u(k-k')$ is the Fourier transform of the interaction energy $u(\mathbf{r})$ between an electron and a single impurity located at the origin, i.e.,

$$u(k-k')=\delta_{\sigma\sigma'}\int d^3r\, e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}u(\mathbf{r}). \tag{1.4}$$

Without loss of generality we shall take $u(\mathbf{q}=0)=0$.

Generalizing the driving field of SA-I to be an arbitrary electromagnetic field of wave vector $\mathbf{q}$ and frequency ω (rather than just a longitudinal field), we can find that the steady-state linear response of the density matrix is

$$\Delta\rho(t)=e^{-i\omega t}e\mathcal{E}_{q\omega}{}^\alpha\int_0^\infty d\tau \times e^{i\omega\tau}\int_0^\beta d\gamma e^{-iH\tau}\rho_0 e^{\gamma H}J_{-q}{}^\alpha e^{-\gamma H}e^{iH\tau} -e^{-i\omega t}(-e/c)A_{q\omega}{}^\alpha\int_0^\beta d\gamma\rho_0 e^{\gamma H}J_{-q}{}^\alpha e^{-\gamma H}. \tag{1.5}$$

Here, $J_{-q}{}^\alpha$ is the Fourier transform of the particle current density operator, as in SA-I (2.13), and ρ_0 is the thermal equilibrium density matrix. Note that the first term of (1.5) is given in terms of the electric field $\mathcal{E}_{q\omega}{}^\alpha$, whether transverse or longitudinal, while the second term is expressed in terms of the vector potential $A_{q\omega}{}^\alpha$. In order to prove (1.5), one has to use the Kubo identity[16] and integrate by parts.[17] For the observables of interest, namely, the induced particle and current densities, it is not difficult to prove that the contribution of the second term in (1.5), which does not involve a "time relaxation," can be expressed in terms of the magnetic field alone and is usually extremely small. Thus, only the first term of (1.5), which is the most difficult to evaluate, will be considered in the following.

[16] R. Kubo, J. Phys. Soc. Japan **12**, 570 (1957).

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As was shown in SA-I, the induced steady-state value of an observable

$$A=\sum_i a(i) =\sum_k\sum_{q'}\langle k-\tfrac{1}{2}q'|a|k+\tfrac{1}{2}q'\rangle c_{k-q'/2}{}^\dagger c_{k+q'/2} \tag{1.6}$$

of the system for a random distribution of the impurities, is given by

$$\bar{A}(q\omega)=\sum_k\langle k-\tfrac{1}{2}q|a|k+\tfrac{1}{2}q\rangle f_{q\omega}(k), \tag{1.7}$$

where the distribution function $f_{q\omega}(k)$, averaged over the random impurities, is defined as

$$f_{q\omega}(k)=F_{q\omega}{}^\alpha\int_0^\infty d\tau e^{i\omega\tau}\int_0^\beta d\gamma \times\langle\mathrm{Tr}\{\rho_0 e^{\gamma H}J_{-q}{}^\alpha e^{-\gamma H}e^{iH\tau}c_{k-q/2}{}^\dagger c_{k+q/2}e^{-iH\tau}\}\rangle_i, \tag{1.8}$$

just as in SA-I (2.20), where now, however, $F_{q\omega}{}^\alpha=e\mathcal{E}_{q\omega}{}^\alpha$, the electric field being transverse or longitudinal. In (1.8), ω is understood to have a small positive imaginary part.

It was shown in SA-I that this distribution function obeys the transport equation SA-I (4.3), namely,

$$i(\epsilon_{k+q/2}-\epsilon_{k-q/2}-\omega)f_{q\omega}(k) =\sum_{k'}W_{q\omega}(kk')f_{q\omega}(k')+D_{q\omega}(k). \tag{1.9}$$

The coefficients of this transport equation, namely, the collision and driving functions $W_{q\omega}(kk')$ and $D_{q\omega}(k)$, respectively, are expressed in terms of the contributions of diagrams, which are described in detail in SA-I. According to the rules derived in SA-I, these coefficients can be evaluated in powers of the impurity density n_i by summing the contributions of the corresponding diagrams that have (for first order in n_i) one bunch of impurity lines originating from a single point, (for second order in n_i) two bunches originating from two points, and so on. Thus, this diagrammatic technique provides a simple method for identifying the contributions to the coefficients $W_{q\omega}(kk')$ and $D_{q\omega}(k)$ that are proportional to n_i, n_i^2, etc. The technical problem still remains, however, of summing the contributions of all relevant diagrams. We shall show that this can be done with relative simplicity.

In Sec. II we carry out the evaluation of the coefficients $W_{q\omega}(kk')$ and $D_{q\omega}(k)$ to first order in n_i in terms of the t matrix[18] of a single impurity. We emphasize that this evaluation is valid for arbitrary $\mathbf{q}$ and ω and for an arbitrarily strong one-impurity potential $u(\mathbf{r})$. In order to demonstrate how the technique works to second

[18] See, e.g., Goldberger and Watson, *Collision Theory* (Wiley-Interscience, Inc., New York, 1964).

order in n_i, we give the evaluation of $W_{q\omega}(kk')$ to that order in Sec. III. Finally, in Sec. IV we examine the limiting case $(\omega,\mathbf{q})\rightarrow 0$ and the results are compared with those of Luttinger and Kohn.[2]

II. EVALUATION OF $W_{q\omega}(kk')$ AND $D_{q\omega}(k)$ TO FIRST ORDER IN n_i

We first evaluate the sum $W_{q\omega}^{[1]}(kk')$ of the contributions of all diagrams for the collision function $W_{q\omega}(kk')$ that are proportional to the impurity density n_i. According to the rules in SA-I, the diagrams for $W_{q\omega}^{[1]}(kk')$ are of the general form shown in Figs. 1 and 2; in each of these, all impurity lines belong to a single bunch. The diagrams of Fig. 1 are characterized by the fact that one of the fermion lines runs from one end to the other of the real time axis without any impurity vertices on it. Their contribution to $W_{q\omega}^{[1]}(kk')$ will be denoted by $W_{q\omega}^{[1A]}(kk')$, which is clearly proportional to $\delta_{kk'}$. The diagrams of Fig. 2, however, have impurity vertices distributed between both fermion lines; their contribution will be denoted by $W_{q\omega}^{[1B]}(kk')$.

In this application it proves convenient to represent by a single diagram, like the one in Fig. 1(a), the sum of all diagrams that differ from it only in that some of the impurity vertices on the real time axis are displaced above or below that axis. (As in SA-I, we do not explicitly display the real time axis in a shorthand diagram.) Note that this shorthand differs from that of SA-I in that "conjugate" diagrams, i.e., those obtained from each other by reversing the directions of all the fermion lines while moving all real time vertices above (below) the real axis below (above) it, are considered separately. Thus the diagram in Fig. 1(b) is the conjugate of that in Fig. 1(a) and is to be evaluated separately. We use this convention, because, in the case of nonvanishing $\mathbf{q}$ and ω that we are studying, the contributions of "conjugate" diagrams are *not* simply the complex conjugates of each other. This shorthand proves extremely convenient in this case of no interparticle interaction, because, as is easily verified from the rules given in SA-I, the contribution of a shorthand diagram (i.e., the sum of the contributions of all diagrams that it represents) is given by the contribution

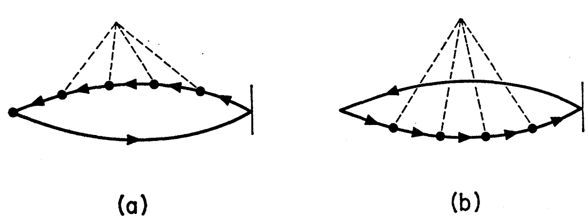

FIG. 1. Diagrams for the part $W_{q\omega}^{[1A]}(kk')$ of the first-order collision function $W_{q\omega}^{[1]}(kk')$, which is proportional to $\delta_{kk'}$. We use shorthand diagrams in which no real time axes are displayed. A shorthand diagram represents the sum of all diagrams that look like it, except that the positions of the vertices with respect to the real time axis in each of them is displayed. Note that diagrams (a) and (b) are "conjugate" to each other.

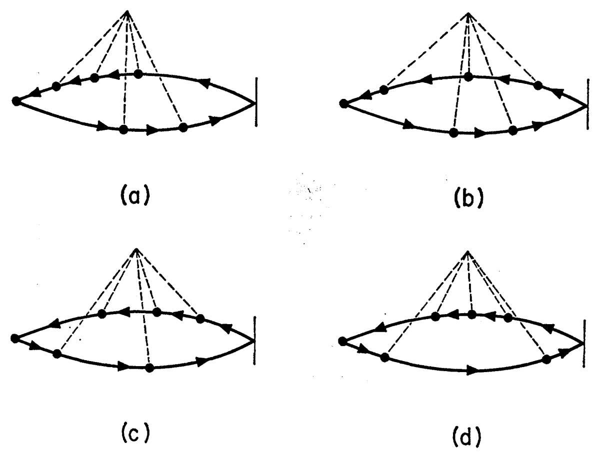

FIG. 2. Diagrams for $W_{q\omega}^{[1B]}(kk')$. In diagrams (a) and (b), the impurity interaction vertex farthest to the left is on the upper line (going from right to left) while it is on the lower line in diagrams (c) and (d). The vertex farthest to the right is on the upper line in diagrams (b) and (c) and on the lower line in (a) and (d).

of the single diagram displayed with the statistical factors that are associated with the fermion lines of the diagram ignored altogether.

The sum of all diagrams (i.e., with any number of impurity lines) of the type shown in Fig. 1 is

$$W_{q\omega}^{[1A]}(kk')$$
$$=\delta_{kk'}n_i\sum_{n=2}^{\infty}\sum_{k_1}\cdots\sum_{k_{n-1}}u(k-k_1)u(k_1-k_2)\cdots u(k_{n-1}-k)$$
$$\times\Big\{(-i)^n\prod_{j=1}^{n-1}[i(\epsilon_{k_j+q/2}-\epsilon_{k-q/2}-\omega)]^{-1}$$
$$+i^n\prod_{j=1}^{n-1}[i(\epsilon_{k+q/2}-\epsilon_{k_j-q/2}-\omega)]^{-1}\Big\},\quad(2.1)$$

the first (second) term being the contribution of all diagrams of the type shown in Fig. 1(a) [1(b)]. In (2.1), $k_1+q/2,\ldots,k_{n-1}+q/2$ label, from left to right, the momenta of the fermion lines between the impurity vertices in a diagram like Fig. 1(a) with n impurity lines [a similar statement is true for Fig. 1(b)]. The factor $\delta_{kk'}$ in (2.1) makes it clear that $W_{q\omega}^{[1A]}(kk')$ represents the usual "scattering-out" part of the collision function $W_{q\omega}^{[1]}(kk')$.

In order to write (2.1) in a more meaningful form, we introduce the quantities $g_k(z)$ and $t_{kk'}(z)$, defined as

$$g_k(z)=(z-\epsilon_k)^{-1},\quad(2.2)$$

$$t_{kk'}(z)=u(k-k')$$
$$+\sum_{n=2}^{\infty}\sum_{k_1}\cdots\sum_{k_{n-1}}u(k-k_1)\cdots u(k_{n-1}-k')$$
$$\times\prod_{j=1}^{n-1}g_{k_j}(z),\quad(2.3)$$

where z is a complex variable. As it is given above, $t_{kk'}(z)$ is precisely the perturbation expansion of the well-known t matrix[18] for a particle scattering off a single impurity located at the origin, i.e., off a potential $u(\mathbf{r})$. It is easily seen that (2.3) is the iteration of either one of the standard t-matrix equations

$$t_{kk'}(z)=u(k-k')+\sum_{k''} u(k-k'')g_{k''}(z)t_{k''k'}(z)$$
$$=u(k-k')+\sum_{k''} t_{kk''}(z)g_{k''}(z)u(k''-k'). \quad (2.4)$$

We note that both $g(z)$ and $t(z)$ are analytic off the real axis; $g_k(z)$ has a pole at ϵ_k, while $t_{kk'}(z)$, in general, has a branch cut along the positive real axis (in the limit of infinite volume) and poles on the negative real axis for possible bound states. Furthermore, as $z\rightarrow\infty$,

$$t_{kk'}(z)-u(k-k')\sim 1/z, \quad (2.5)$$

as can be seen from (2.3). Here we demonstrate a useful relation between the difference and the product of two t matrices, namely,

$$t_{k+q/2,k'+q/2}(z+\tfrac{1}{2}z')-t_{k-q/2,k'-q/2}(z-\tfrac{1}{2}z')$$
$$=\sum_{k''} t_{k+q/2,k''+q/2}(z+\tfrac{1}{2}z') \times G_{k''q}(z,z')t_{k''-q/2,k'-q/2}(z-\tfrac{1}{2}z')$$
$$=\sum_{k''} t_{k-q/2,k''-q/2}(z-\tfrac{1}{2}z') \times G_{k''q}(z,z')t_{k''+q/2,k'+q/2}(z+\tfrac{1}{2}z'), \quad (2.6)$$

where

$$G_{kq}(z,z')=g_{k+q/2}(z+\tfrac{1}{2}z')-g_{k-q/2}(z-\tfrac{1}{2}z'). \quad (2.7)$$

To prove (2.6), we note that its left-hand side is, according to (2.3),

$$\sum_{n=2}^{\infty}\sum_{k_1}\cdots\sum_{k_{n-1}} u(k-k_1)\cdots u(k_{n-1}-k') \times\Big[\prod_{i=1}^{n-1} g_{k_i+q/2}(z+\tfrac{1}{2}z')-\prod_{i=1}^{n-1} g_{k_i-q/2}(z-\tfrac{1}{2}z')\Big]. \quad (2.8)$$

If we now apply the formula

$$\prod_{i=1}^{n} x_i-\prod_{i=1}^{n} y_i=\sum_{j=1}^{n}\Big(\prod_{i=1}^{j-1} x_i\Big)(x_j-y_j)\Big(\prod_{i=j+1}^{n} y_i\Big) \quad (2.9)$$

to the difference of products in (2.8), rearrange the sums, and relabel the momentum variables, we get the first equality of (2.6); the second equality is a trivial consequence of the first.

We can now write $W_{q\omega}{}^{[1A]}(kk')$ in terms of the t matrix by comparing (2.1) and (2.3); thus,

$$W_{q\omega}{}^{[1A]}(kk')=\delta_{kk'}n_i(1/i)\big[t_{k+q/2,k+q/2}(\epsilon_{k-q/2}+\omega) -t_{k-q/2,k-q/2}(\epsilon_{k+q/2}-\omega)\big]. \quad (2.10)$$

Furthermore, since $t(z)$ is analytic off the real axis and behaves at infinity according to (2.5), we can rewrite (2.10) as an integral:

$$W_{q\omega}{}^{[1A]}(kk')=\delta_{kk'}n_i\int_{-\infty}^{+\infty}\frac{dz}{2\pi}G_{kq}(z,\omega) \times\big[t_{k+q/2,k+q/2}(z+\tfrac{1}{2}\omega)-t_{k-q/2,k-q/2}(z-\tfrac{1}{2}\omega)\big], \quad (2.11)$$

where $G_{kq}(z,\omega)$ is given by (2.7). To see this, we recall that ω has a positive imaginary part and we close the contour in (2.11) in the upper (lower) half-plane for the first (second) term, thereby enclosing the pole of $G_{kq}(z,\omega)$ at $\epsilon_{k-q/2}+\frac{1}{2}\omega(\epsilon_{k+q/2}-\frac{1}{2}\omega)$. Finally, if we use the identity (2.6), we obtain for $W_{q\omega}{}^{[1A]}(kk')$ the equivalent expressions

$$W_{q\omega}{}^{[1A]}(kk')=-\delta_{kk'}n_i\sum_{k''} w_{q\omega}{}^{(1)}(kk'')$$
$$=-\delta_{kk'}n_i\sum_{k''} w_{q\omega}{}^{(1)}(k''k), \quad (2.12)$$

where

$$w_{q\omega}{}^{(1)}(kk')=-\int_{-\infty}^{+\infty}\frac{dz}{2\pi}G_{kq}(z,\omega) \times T_{kk',q}(z,\omega)G_{k'q}(z,\omega) \quad (2.13)$$

and

$$T_{kk',q}(z,\omega)\equiv t_{k+q/2,k'+q/2}(z+\tfrac{1}{2}\omega) \times t_{k'-q/2,k-q/2}(z-\tfrac{1}{2}\omega). \quad (2.14)$$

The expression (2.12) suggests that $n_iw_{q\omega}{}^{(1)}(kk')$ plays the role of a scattering probability rate from the state k' to the state k in the case of an arbitrarily varying electric field. That this is the significance of $n_iw_{q\omega}{}^{(1)}(kk')$ is substantiated further by the study carried out below of $W_{q\omega}{}^{[1B]}(kk')$, the part of the collision function that represents the "scattering-in."

Typical diagrams for $W_{q\omega}{}^{[1B]}(kk')$ are shown in Figs. 2(a)–2(d). These four types of diagrams are characterized as follows: The impurity vertex farthest to the left is found on the upper fermion line (directed to the left) in diagrams 2(a) and 2(b), and on the lower fermion line (directed to the right) in diagrams 2(c) and 2(d). The impurity vertex farthest to the right is on the upper fermion line in diagrams 2(b) and 2(c), while it is on the lower fermion line in diagrams 2(a) and 2(d). We shall evaluate here only the contribution of all the diagrams (i.e., with any number of impurity lines) of the type shown in Fig. 2(a), to be denoted by $W_{q\omega}{}^{(2a)}(kk')$. The other contributions $W_{q\omega}{}^{(2b)}(kk')$, $W_{q\omega}{}^{(2c)}(kk')$, $W_{q\omega}{}^{(2d)}(kk')$ are evaluated in essentially the same way, and we shall therefore simply incorporate these into the final result. Applying the rules of SA-I for the evaluation of $W_{q\omega}(kk')$ to the diagrams of the type 2(a), we have

$$W_{q\omega}^{(2a)}(kk')=n_i\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}(-i)^n(i)^l(1/i)^{n+l-1}$$
$$\times\sum_{k_1}\cdots\sum_{k_n}\sum_{p_0}\cdots\sum_{p_{l-1}}\delta_{k_n,k'+q/2}\delta_{p_0,k-q/2}$$
$$\times u(k+\tfrac{1}{2}q-k_1)\cdots u(k_n-k'-\tfrac{1}{2}q)$$
$$\times u(k'-\tfrac{1}{2}q-p_{l-1})\cdots u(p_1-k+\tfrac{1}{2}q)$$
$$\times P(\epsilon_{k_1}\cdots\epsilon_{k_n},\epsilon_{p_0}\cdots\epsilon_{p_{l-1}},\omega)\,,\quad(2.15)$$

where

$$P(\epsilon_{k_1}\cdots\epsilon_{k_n},\epsilon_{p_0}\cdots\epsilon_{p_{l-1}},\omega)$$
$$\equiv\sum_{\nu_0=1}^{n}\sum_{\nu_1=\nu_0}^{n}\cdots\sum_{\nu_{l-2}=\nu_{l-3}}^{n}\prod_{s=0}^{l-1}\Big[\prod_{j=\nu_{s-1}}^{\nu_s}(\epsilon_{k_j}-\epsilon_{p_s}-\omega)^{-1}\Big],\quad(2.16)$$

with $\nu_{-1}\equiv 1$ and $\nu_{l-1}\equiv n$. In (2.15), n stands for the number of impurity vertices on the upper fermion line and l for the number on the lower fermion line; $k_1, k_2, \ldots, k_n=k'+\frac{1}{2}q$ label the momenta of the upper fermion lines between the impurity vertices, while $p_0=k-\frac{1}{2}q, p_1, \ldots, p_{l-1}$ label the corresponding ones of the lower fermion lines; finally, the factor $P(\epsilon_{k_1}\cdots\epsilon_{k_n},\epsilon_{p_0}\cdots\epsilon_{p_{l-1}},\omega)$ denotes the sum of all the products of the factors $(\Delta\epsilon_j-\omega)^{-1}$ that arise from all possible orderings of the $n+l-2$, as yet unfixed, vertices on the real time axis. This quantity is discussed further in the Appendix, where it is shown [see (A3)] that

$$P(\epsilon_{k_1}\cdots\epsilon_{k_n},\epsilon_{p_0}\cdots\epsilon_{p_{l-1}},\omega)$$
$$=(-1)^n\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\prod_{j=1}^{n}g_{k_j}(z+\tfrac{1}{2}\omega)\prod_{i=0}^{l-1}g_{p_i}(z-\tfrac{1}{2}\omega)\,.\quad(2.17)$$

If we now insert (2.17) in (2.15) and take into account the definitions (2.3) and (2.7), we find for $W_{q\omega}^{(2a)}(kk')$ the simple expression

$$W_{q\omega}^{(2a)}(kk')=n_i\int_{-\infty}^{+\infty}\frac{dz}{2\pi}g_{k-q/2}(z-\tfrac{1}{2}\omega)$$
$$\times T_{kk',q}(z,\omega)g_{k'+q/2}(z+\tfrac{1}{2}\omega)\,.\quad(2.18)$$

The factors $g_{k-q/2}(z-\frac{1}{2}\omega)g_{k'+q/2}(z+\frac{1}{2}\omega)$ in (2.18) come from the fact that in diagram 2(a) the impurity line farthest to the left (right) is on the upper (lower) fermion line. In a similar way, one obtains for $W_{q\omega}^{(2b)}(kk')$, $W_{q\omega}^{(2c)}(kk')$, $W_{q\omega}^{(2d)}(kk')$ integrals like (2.18), with the factors just mentioned replaced by

$$(-1)g_{k-q/2}(z-\tfrac{1}{2}\omega)g_{k'-q/2}(z-\tfrac{1}{2}\omega)\,,$$
$$g_{k+q/2}(z+\tfrac{1}{2}\omega)g_{k'-q/2}(z-\tfrac{1}{2}\omega)\,,$$
$$(-1)g_{k+q/2}(z+\tfrac{1}{2}\omega)g_{k'+q/2}(z+\tfrac{1}{2}\omega)\,,$$

respectively. Putting all these results together, we get

$$W_{q\omega}^{[1B]}(kk')=-n_i\int_{-\infty}^{+\infty}\frac{dz}{2\pi}G_{kq}(z,\omega)T_{kk',q}(z,\omega)G_{k'q}(z,\omega)$$
$$=n_i w_{q\omega}^{(1)}(kk')\,,\quad(2.19)$$

the last equality following from definition (2.13).

Combining the results (2.12) and (2.19), we have that the collision term of the general transport equation (1.9) is, to the first order in n_i,

$$\sum_{k'}W_{q\omega}^{[1]}(kk')f_{q\omega}(k')$$
$$=\sum_{k'}n_i[w_{q\omega}^{(1)}(kk')f_{q\omega}(k')$$
$$-w_{q\omega}^{(1)}(k'k)f_{q\omega}(k)]\,,\quad(2.20)$$

where the kernel $w_{q\omega}^{(1)}(kk')$, which plays the role of a transition probability rate, is given by (2.13) in terms of the t matrix for the scattering of a particle off a single impurity located at the origin.

We now turn to the evaluation of the contributions $D_{q\omega}^{[0]}(k)$ and $D_{q\omega}^{[1]}(k)$ of zeroth and first order in the impurity density, respectively, of the driving function $D_{q\omega}(k)$.

The zeroth-order term is simply

$$D_{q\omega}^{[0]}(k)=-F_{q\omega}{}^{\alpha}\frac{k_\alpha}{m}\frac{F(\epsilon_{k+q/2})-F(\epsilon_{k-q/2})}{\epsilon_{k+q/2}-\epsilon_{k-q/2}}\,,\quad(2.21)$$

where $F(\epsilon)$ is the Fermi-Dirac distribution function

$$F(\epsilon)=[e^{\beta(\epsilon-\mu)}+1]^{-1}.\quad(2.22)$$

For later convenience, we note that $F(z)$, as a function of a complex variable, has poles at $z=\mu+(2n+1)\pi i/\beta$ (n an integer), all of which are off the real axis. We can thus represent (2.21) by the integral

$$D_{q\omega}^{[0]}(k)=F_{q\omega}{}^{\alpha}\frac{k_\alpha}{m}\int_C\frac{dz}{2\pi i}F(z)g_{k+q/2}(z)g_{k-q/2}(z)\,,\quad(2.23)$$

where C is a contour made up of one line going from $-\infty$ to $+\infty$ above the real axis and another from $+\infty$ to $-\infty$ below it, as in Fig. 3.

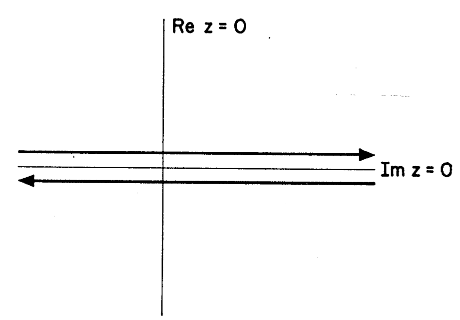

FIG. 3. The contour C used in integrals found in contributions to $D_{q\omega}(k)$. C runs just above the real axis in the positive direction and just below in the negative direction.

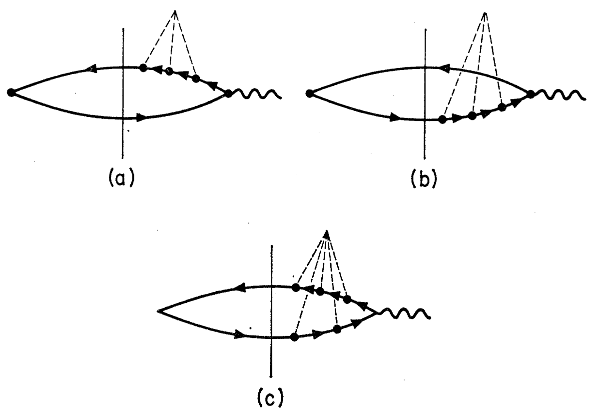

FIG. 4. Diagrams for $D_{q\omega}^{[1A]}(k)$ the part of $D_{q\omega}^{[1]}(k)$ with no explicit dependence on ω. In the shorthand we use here, vertices occurring to the right of the vertical line are in the imaginary time region, while those to the left are in the real time region. The real time axis is not displayed in these diagrams because we are using the same shorthand for real time vertices as in Figs. 1 and 2. Also, we mean to represent by one diagram all those which differ from it only by having the vertices on the imaginary axis in a different relative ordering.

The contributions to $D_{q\omega}^{[1]}(k)$ can be classified according to whether or not, apart from $F_{q\omega}^{\alpha}$, they depend on the frequency ω. The part of $D_{q\omega}^{[1]}(k)$ that has no such dependence, to be denoted by $D_{q\omega}^{[1A]}(k)$, is obtained by evaluating the contributions $D_{q\omega}^{(4a)}(k)$, $D_{q\omega}^{(4b)}(k)$ and $D_{q\omega}^{(4c)}(k)$ of all diagrams of the types depicted in Figs. 4(a), 4(b), and 4(c), respectively. These are diagrams with all impurity vertices to the right of the vertical line, i.e., in the imaginary time region. (As in SA-I, we take a diagram to represent not only itself, but also those which differ from it only by the ordering of the impurity vertices along the imaginary time axis.) By applying the rules described in SA-I, we find that the sum of the contributions of all diagrams of type 4(a) with any number of impurity lines is

$$D_{q\omega}^{(4a)}(k)=F_{q\omega}^{\alpha}\frac{k_\alpha}{m}n_i\sum_{n=1}^{\infty}(-1)^{n+1}\times\sum_{k_1}\cdots\sum_{k_{n-1}}u(k_n-k_{n-1})\cdots u(k_1-k_0)\times Q(\epsilon_{k_{-1}}\epsilon_{k_0}\epsilon_{k_1}\cdots\epsilon_{k_n}),\quad(2.24)$$

where

$$Q(\epsilon_{k_{-1}}\epsilon_{k_0}\epsilon_{k_1}\cdots\epsilon_{k_n})=[-F(\epsilon_{k_n})][1-F(\epsilon_{k_{-1}})]\int_0^\beta d\gamma_0\cdots\int_0^\beta d\gamma_n\times\prod_{i=0}^{n}[\exp\gamma_i(\epsilon_{k_i}-\epsilon_{k_{i-1}})]\times\prod_{j=0}^{n}\{\theta(\gamma_j-\gamma_{j-1})[1-F(\epsilon_{k_{j-1}})]-\theta(\gamma_{j-1}-\gamma_j)F(\epsilon_{k_{j-1}})\},\quad(2.25)$$

with $k_{-1}\equiv k-\frac{1}{2}q$ and $k_0=k_n\equiv k+\frac{1}{2}q$. In (2.25) we have introduced for convenience the unit step function $\theta(x)$, which is equal to 1 for $x>0$ and equal to 0 for $x<0$. In analogy to the quantity P in (2.17), the quantity Q can be expressed simply as

$$Q(\epsilon_{k_{-1}}\epsilon_{k_0}\cdots\epsilon_{k_n})=(-1)^{n+1}\int_C\frac{dz}{2\pi i}F(z)g_{k+q/2}(z)[\prod_{i=1}^{n-1}g_{k_i}(z)]\times g_{k+q/2}(z)g_{k-q/2}(z),\quad(2.26)$$

with the contour C as in Fig. 3. This result is proven in the Appendix [see (A9)]. Now using the definition (2.4) for $t_{kk'}(z)$ in conjunction with (2.26), we find from (2.24) that

$$D_{q\omega}^{(4a)}(k)=n_iF_{q\omega}^{\alpha}\frac{k_\alpha}{m}\int_C\frac{dz}{2\pi i}F(z)g_{k+q/2}(z)\times t_{k+q/2,k+q/2}(z)g_{k+q/2}(z)g_{k-q/2}(z).\quad(2.27)$$

In a similar way, we find that

$$D_{q\omega}^{(4b)}(k)=n_iF_{q\omega}^{\alpha}\frac{k_\alpha}{m}\int_C\frac{dz}{2\pi i}F(z)g_{k+q/2}(z)\times g_{k-q/2}(z)t_{k-q/2,k-q/2}(z)g_{k-q/2}(z)\quad(2.28)$$

and

$$D_{q\omega}^{(4c)}(k)=n_iF_{q\omega}^{\alpha}\sum_{k'}\frac{k_\alpha'}{m}\int_C\frac{dz}{2\pi i}\times F(z)g_{k+q/2}(z)t_{k+q/2,k'+q/2}(z)g_{k'+q/2}(z)\times g_{k'-q/2}(z)t_{k'-q/2,k-q/2}(z)g_{k-q/2}(z).\quad(2.29)$$

Collecting these results, we can write

$$D_{q\omega}^{[1A]}(k)=n_i\Delta_q(kk),\quad(2.30)$$

where the more general quantity $\Delta_q(kk')$, which will be useful later, is defined by

$$\Delta_q(kk')=F_{q\omega}^{\alpha}\int_C\frac{dz}{2\pi i}F(z)\times g_{k+q/2}(z)\bigg(\frac{k_\alpha}{m}g_{k-q/2}(z)t_{k-q/2,k'-q/2}(z)+\sum_{k''}t_{k+q/2,k''+q/2}(z)g_{k''+q/2}(z)\frac{k_\alpha''}{m}\times g_{k''-q/2}(z)t_{k''-q/2,k'-q/2}(z)+t_{k+q/2,k'+q/2}(z)g_{k'+q/2}(z)\frac{k_\alpha'}{m}\bigg)g_{k'-q/2}(z).\quad(2.31)$$

It is worthwhile to note that this rather complicated expression simplifies considerably for the case of a longitudinal force field $F_{q\omega}{}^{\alpha} \equiv -iq^{\alpha}\phi_{q\omega}$. Using the identity (2.6) and the relation $q^{\alpha}k_{\alpha}/m = \epsilon_{k+q/2} - \epsilon_{k-q/2}$, we find

$$\Delta_q(kk') = -\phi_{q\omega}\int_C \frac{dz}{2\pi}F(z) \times[g_{k+q/2}(z)t_{k+q/2,k'+q/2}(z)g_{k'+q/2}(z) - g_{k-q/2}(z)t_{k-q/2,k'-q/2}(z)g_{k'-q/2}(z)]. \quad (2.32)$$

We now consider the contribution $D_{q\omega}{}^{[1B]}(k)$ to $D_{q\omega}{}^{[1]}(k)$ which depends on the frequency ω. Examples of the diagrams that contribute to $D_{q\omega}{}^{[1B]}(k)$ are shown in Fig. 5; these diagrams have some impurity vertices on the left of the vertical line, i.e., in the real time region. Actually these diagrams are meant to represent also those that differ from them, in that the impurity vertices lying to the right of the vertical line (i.e., in the imaginary time region) lie either all on the bottom line or both on the top and the bottom lines. Diagram 5(c) differs from 5(d) in that the vertex farthest to the left lies on the lower line in 5(c) and on the upper line in 5(d).

We shall not give here a detailed exposition of the evaluation of these diagrams; instead, we sketch the procedure. To handle the factors arising from the vertices and intermediate states in the real time region (to the left of the vertical line in a diagram), we apply the methods developed to evaluate the contribution of the diagrams in Figs. 1 and 2. The techniques used for the evaluation of the diagrams in Fig. 4 are used to obtain the contribution of the part of any one of these diagrams in the imaginary time region (to the right of the vertical line). We note that this part of the evaluation always yields a factor $\Delta_q(k'k'')$. Rather than displaying the result for $D_{q\omega}{}^{[1B]}(k)$ alone, it is convenient to combine it with (2.30); thus we have altogether

$$D_{q\omega}{}^{[1]}(k) = D_{q\omega}{}^{[1A]}(k) + D_{q\omega}{}^{[1B]}(k) = -n_i\sum_{k'k''}\int_{-\infty}^{\infty}\frac{dz}{2\pi i}G_{kq}(z,\omega) \times[\delta_{kk'} + t_{k+q/2,k'+q/2}(z+\tfrac{1}{2}\omega)g_{k'+q/2}(z+\tfrac{1}{2}\omega)] \times[\delta_{k''k} + g_{k''-q/2}(z-\tfrac{1}{2}\omega)t_{k''-q/2,k-q/2}(z-\tfrac{1}{2}\omega)] \times\Delta_q(k'k''). \quad (2.33)$$

The term in (2.33) associated with $\delta_{kk'}\delta_{kk''}$ is precisely $D_{q\omega}{}^{[1A]}(k)$, as given by (2.30). This term is simply the modification (to first order in n_i) of the driving term $D_{q\omega}{}^{[0]}(k)$ [see (2.21)] due to the fact that the electrons were in thermal equilibrium in the presence of the impurities before the electric field was turned on. The remaining terms in (2.33) give $D_{q\omega}{}^{[1B]}(k)$, which describes (to first order in n_i) the effects of the electric field on the collisions of the electrons with the impurities, i.e., $D_{q\omega}{}^{[1B]}(k)$ is the interference term.

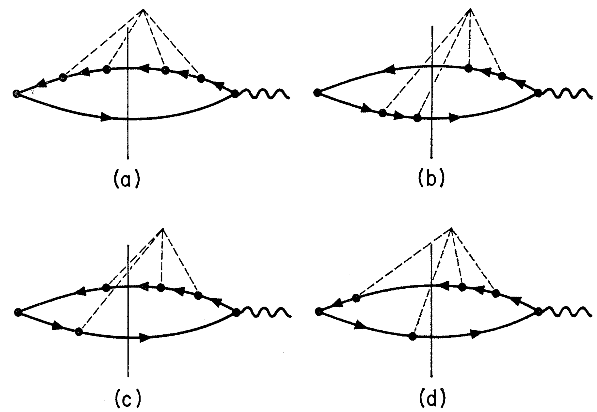

FIG. 5. Diagrams for $D_{q\omega}{}^{[1B]}(k)$, which depends explicitly on ω. Here we do not display all the possible positions that the external field vertex (represented by the wiggly line) can have with respect to the impurity vertices in the imaginary time region.

Thus, Eqs. (2.20), (2.13), (2.21), (2.33), and (2.31) give the coefficients of the transport equation (1.9) for arbitrary **q** and ω in terms of the t matrix for the scattering of an electron from a simple impurity at the origin, but up to the first order in the impurity density n_i. These coefficients are given above, primarily for convenience, in terms of integrals; these can easily be carried out by contour integration.

III. EVALUATION OF $W_{q\omega}(kk')$ TO SECOND ORDER IN n_i

In this section we calculate $W_{q\omega}{}^{[2]}(kk')$, the contribution of second order in n_i to the collision function $W_{q\omega}(kk')$. We do not, however, calculate here the driving function $D_{q\omega}(k)$ to this order since no new techniques are necessary and the calculation is fairly long.

Let us begin by considering a special class of diagrams for $W_{q\omega}{}^{[2]}(kk')$ not proportional to $\delta_{kk'}$, which have one of the two following properties (we note that a second-order diagram cannot have both of these properties): (1) Aside from the fermion lines closing up the diagram at either end, there is one other line in the diagram which has momentum $k\pm\frac{1}{2}q$ or $k'\pm\frac{1}{2}q$. (2) In the diagram there is a pair of lines for which the momentum difference is constrained to be q while the momentum average is unrestricted. The reason we single out these diagrams is because we must be careful, in calculating their contributions, not to include any terms which have factors $(\epsilon_{k''+q/2} - \epsilon_{k''-q/2} - \omega)^{-1}$; a term of this sort cannot, according to the rules discussed in SA-I, occur in $W_{q\omega}(kk')$. Diagrams of this class are shown in Figs. 6–8.

Consider first the diagrams shown in Fig. 6. Diagram 6(a) is distinguished from 6(b) in that in the former, a vertex from the bunch B is farthest to the left, while in the latter a vertex from bunch A is farthest to the left. (In these diagrams we do not make a distinction as to

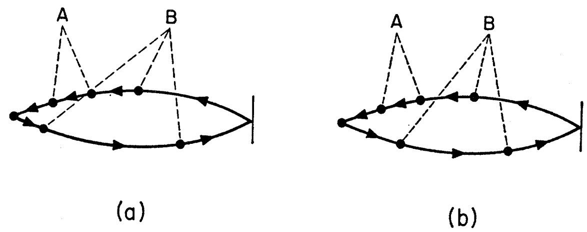

FIG. 6. Diagrams for the contribution $W_{q\omega}^{[6]}(kk')$ to the second-order collision function $W_{q\omega}^{[2]}(kk')$. These diagrams have two fermion lines with momentum $k+\frac{1}{2}q$. If the A bunch were on the lower line there would be two lines with momentum $k-\frac{1}{2}q$.

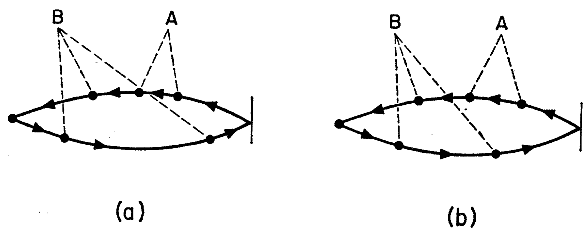

FIG. 7. Diagrams for $W_{q\omega}^{[7]}(kk')$. These diagrams have two fermion lines with momentum $k'+\frac{1}{2}q$.

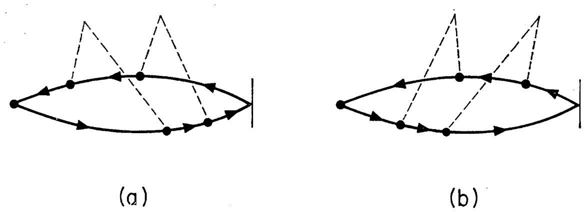

FIG. 8. Diagrams for $W_{q\omega}^{[8]}(kk')$. These diagrams have a pair of fermion lines, the difference of the momenta of which is fixed to be q. The average momentum of these two lines is unrestricted.

whether the vertex farthest to the right is on the upper or lower line.) Note that both of these diagrams have the feature that at least one B vertex is to the left of some A vertices—a diagram not having this feature would not be appropriate, according to the rules given in SA-I. The contribution $W_{q\omega}^{(6a)}(kk')$ of diagrams of the type 6(a) is easily evaluated using the methods evolved in Sec. II and in the Appendix; thus

$$W_{q\omega}^{(6a)}(kk') = -n_i^2\int_{-\infty}^{\infty}\frac{dz}{2\pi}[g_{k+q/2}(z+\tfrac{1}{2}\omega)]^2 t_{k+q/2,k+q/2}(z+\tfrac{1}{2}\omega) \times T_{kk',q}(z,\omega)G_{k'q}(z,\omega). \quad (3.1)$$

To evaluate the contribution of diagrams of the type 6(b), we treat separately the parts of the diagram to the left and the right of the last B vertex on the left. We can treat each such part essentially as described in Sec. II, and we are led to introduce two integrations along the real frequency axis; thus

$$W_{q\omega}^{(6b)}(kk') = -n_i^2\int_{-\infty}^{\infty}\frac{dzdz'}{(2\pi)^2 i} \times g_{k-q/2}(z'-\tfrac{1}{2}\omega)\sum_{k''} t_{k+q/2,k''+q/2}(z'+\tfrac{1}{2}\omega) \times g_{k''+q/2}(z'+\tfrac{1}{2}\omega)g_{k''+q/2}(z+\tfrac{1}{2}\omega) \times t_{k''+q/2,k+q/2}(z+\tfrac{1}{2}\omega)g_{k+q/2}(z+\tfrac{1}{2}\omega) \times T_{kk',q}(z,\omega)G_{k'q}(z,\omega). \quad (3.2)$$

Appealing to (2.6), we can rewrite this as

$$W_{q\omega}^{(6b)}(kk') = -n_i^2\int_{-\infty}^{\infty}\frac{dzdz'}{(2\pi)^2 i}g_{k-q/2}(z'-\tfrac{1}{2}\omega) \times \frac{t_{k+q/2,k+q/2}(z'+\frac{1}{2}\omega)-t_{k+q/2,k+q/2}(z+\frac{1}{2}\omega)}{z-z'} \times g_{k+q/2}(z+\tfrac{1}{2}\omega)T_{kk',q}(z,\omega)G_{k'q}(z,\omega). \quad (3.3)$$

We note here that as a function of z', the fraction in (3.3) is analytic in the upper half-plane; in particular, it is analytic at $z=z'$. Closing the z' contour above and adding in the result (3.1), we get

$$W_{q\omega}^{(6a)}(kk')+W_{q\omega}^{(6b)}(kk') = -n_i^2\int_{-\infty}^{\infty}\frac{dz}{2\pi}G_{kq}(z,\omega)G_{k'q}(z,\omega) \times t_{k+q/2,k+q/2}(z+\tfrac{1}{2}\omega)g_{k+q/2}(z+\tfrac{1}{2}\omega)T_{kk',q}(z,\omega) - n_i^2\int_{-\infty}^{\infty}\frac{dz}{2\pi}t_{k+q/2,k+q/2}(\epsilon_{k-q/2}+\omega) \times g_{k-q/2}(z-\tfrac{1}{2}\omega)g_{k+q/2}(z+\tfrac{1}{2}\omega) \times T_{kk',q}(z,\omega)G_{k'q}(z,\omega). \quad (3.4)$$

Adding to this the result of evaluating diagrams like the ones in Fig. 6, except that the A bunch is on the lower line, we get a contribution

$$W_{q\omega}^{(6)}(kk') = -n_i^2\int_{-\infty}^{\infty}\frac{dz}{2\pi}G_{kq}(z,\omega)G_{k'q}(z,\omega) \times [g_{k+q/2}(z+\tfrac{1}{2}\omega)t_{k+q/2,k+q/2}(z+\tfrac{1}{2}\omega) + g_{k-q/2}(z-\tfrac{1}{2}\omega)t_{k-q/2,k-q/2}(z-\tfrac{1}{2}\omega)] \times T_{kk',q}(z,\omega) + \frac{n_i^2}{i(\epsilon_{k+q/2}-\epsilon_{k-q/2}-\omega)} \times \sum_{k''} w_{q\omega}^{(1)}(kk'')w_{q\omega}^{(1)}(kk'), \quad (3.5)$$

where we have used (2.10), (2.12), and (2.19) to obtain the second term on the right-hand side of (3.5). We remark here that although the second term [with a factor $(\epsilon_{k+q/2}-\epsilon_{k-q/2}-\omega)^{-1}$] is singular in the limit $q,\omega\rightarrow 0$, the whole expression for $W_{q\omega}^{(6)}(kk')$ is not, since the first term has an equal and opposite singular contribution. We have written $W_{q\omega}^{(6)}(kk')$ in this way so that we can make a comparison with the results of Luttinger and Kohn.[2] In a precisely similar way as above, we can evaluate the contributions of the diagrams of Fig. 7 as well as those which differ from them by having their A bunch on the bottom line. We obtain

$$W_{q\omega}^{(7)}(kk')=-n_i^2\int_{-\infty}^{\infty}\frac{dz}{2\pi}G_{kq}(z,\omega)G_{k'q}(z,\omega)T_{kk',q}(z,\omega)\times[g_{k'+q/2}(z+\tfrac{1}{2}\omega)t_{k'+q/2,k'+q/2}(z+\tfrac{1}{2}\omega)+g_{k'-q/2}(z-\tfrac{1}{2}\omega)t_{k'-q/2,k'-q/2}(z-\tfrac{1}{2}\omega)]+\frac{n_i^2}{i(\epsilon_{k'+q/2}-\epsilon_{k'-q/2}-\omega)}\times\sum_{k''}w_{q\omega}^{(1)}(kk')w_{q\omega}^{(1)}(k'k'').\quad(3.6)$$

The last types of diagrams that belong to the special class described earlier are shown in Fig. 8. The essential features of the evaluation of the contribution of these diagrams are like those used for diagrams 6 and 7. After some manipulation, we obtain

$$W_{q\omega}^{(8)}(kk')=-n_i^2\sum_{k''}\int_{-\infty}^{\infty}\frac{dz}{2\pi}G_{kq}(z,\omega)T_{kk'',q}(z,\omega)\times g_{k''+q/2}(z+\tfrac{1}{2}\omega)g_{k''-q/2}(z-\tfrac{1}{2}\omega)T_{k''k',q}(z,\omega)\times G_{k'q}(z,\omega)-\sum_{k''}\frac{n_i^2}{i(\epsilon_{k''+q/2}-\epsilon_{k''-q/2}-\omega)}\times w_{q\omega}^{(1)}(kk'')w_{q\omega}^{(1)}(k''k').\quad(3.7)$$

Note that the terms in (3.6) and (3.7) explicitly involving $w_{q\omega}^{(1)}$ vanish when added together and summed over k'.

Consider now the diagrams for $W_{q\omega}^{[2]}(kk')$ (not proportional to $\delta_{kk'}$) that are *not* members of the class described above. We note that they present no problems similar to the ones discussed above and their contribution can be evaluated exactly as in Sec. II. Denoting it by $W_{q\omega}^{(a)}(kk')$, we shall give here only the result

$$W_{q\omega}^{(a)}(kk')=-n_i^2\int_{-\infty}^{\infty}\frac{dz}{2\pi}G_{kq}(z,\omega)G_{k'q}(z,\omega)\tfrac{1}{2}L_{kk',q}(z,\omega),\quad(3.8)$$

where

$$L_{kk',q}(z,\omega)=\int d^3R_1\int d^3R_2\{\Gamma_{k+q/2,k'+q/2}{}^{12}(z+\tfrac{1}{2}\omega)\Gamma_{k'-q/2,k-q/2}{}^{12}(z-\tfrac{1}{2}\omega)+2\Gamma_{k+q/2,k'+q/2}{}^{12}(z+\tfrac{1}{2}\omega)\times[\tau_{k'-q/2,k-q/2}{}^1(z-\tfrac{1}{2}\omega)+\sum_{k''}\tau_{k'-q/2,k''-q/2}{}^1(z-\tfrac{1}{2}\omega)g_{k''-q/2}(z-\tfrac{1}{2}\omega)\tau_{k''-q/2,k-q/2}{}^2(z-\tfrac{1}{2}\omega)]+2[\tau_{k+q/2,k'+q/2}{}^1(z+\tfrac{1}{2}\omega)+\sum_{k''}\tau_{k+q/2,k''+q/2}{}^1(z+\tfrac{1}{2}\omega)g_{k''+q/2}(z+\tfrac{1}{2}\omega)\tau_{k''+q/2,k'+q/2}{}^2(z+\tfrac{1}{2}\omega)]\times\Gamma_{k'-q/2,k-q/2}{}^{12}(z-\tfrac{1}{2}\omega)+2\sum_{k''k'''}\tau_{k+q/2,k''+q/2}{}^1(z+\tfrac{1}{2}\omega)g_{k''+q/2}(z+\tfrac{1}{2}\omega)\tau_{k''+q/2,k'+q/2}{}^2(z+\tfrac{1}{2}\omega)\times\tau_{k'-q/2,k'''-q/2}{}^1(z-\tfrac{1}{2}\omega)g_{k'''-q/2}(z-\tfrac{1}{2}\omega)\tau_{k'''-q/2,k-q/2}{}^2(z-\tfrac{1}{2}\omega)\},\quad(3.9)$$

$$\tau_{kk'}{}^j(z)=e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}_j}t_{kk'}(z)=u^j(k-k')+\sum_{k''}u^j(k-k'')g_{k''}(z)\tau_{k''k'}{}^j(z),\quad(3.10)$$

and

$$u^j(k-k')=e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}_j}u(k-k').\quad(3.11)$$

We also have the defining equation

$$\Gamma_{kk'}{}^{12}(z)=\sum_{k''k'''}u^1(k-k'')g_{k''}(z)\tau_{k''k'''}{}^2(z)g_{k'''}(z)\tau_{k'''k'}{}^1(z)+\sum_{k''k'''}u^2(k-k'')g_{k''}(z)\tau_{k''k'''}{}^1(z)g_{k'''}(z)\tau_{k'''k'}{}^2(z)+\sum_{k''}[u^1(k-k'')+u^2(k-k'')]\times g_{k''}(z)\Gamma_{k''k'}{}^{12}(z).\quad(3.12)$$

We note here that, of course, from the diagram technique we do not directly obtain Γ^{12}, but rather its iteration. In Eq. (3.9) the volume integrations are over the impurity positions $\mathbf{R}_1$ and $\mathbf{R}_2$.

Thus, denoting the sum of all contributions to $W_{q\omega}^{[2]}(kk')$ not proportional to $\delta_{kk'}$ by $n_i^2w_q^{(2)}(kk')$, we have

$$n_i^2w_{q\omega}^{(2)}(kk')=W_{q\omega}^{(a)}(kk')+W_{q\omega}^{(6)}(kk')+W_{q\omega}^{(7)}(kk')+W_{q\omega}^{(8)}(kk').\quad(3.13)$$

We now consider briefly the contribution to $W_{q\omega}^{[2]}(kk')$ that is proportional to $\delta_{kk'}$, to be denoted

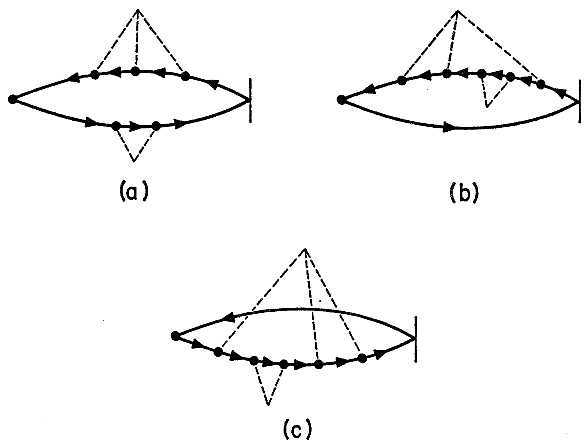

FIG. 9. Some diagrams for the part $W_{q\omega}^{[2A]}(kk')$ of $W_{q\omega}^{[2]}(kk')$ which is proportional to $\delta_{kk'}$. Diagram 9(a) is closely related to the diagrams in Fig. 6. The contribution of diagrams 9(b) and 9(c) has a part closely related to the contributions of the diagrams in Figs. 7 and 8; the remainder is related to $W_{q\omega}^{(a)}(kk')$.

by $W_{q\omega}^{[2A]}(kk')$. In Fig. 9(a) we have displayed the type of diagram that has a contribution connected to $W_{q\omega}^{(6)}(kk')$; in fact, it can be shown that the contribution of diagrams of this type can be written as

$$-\delta_{kk'}\sum_{k''} W_{q\omega}^{(6)}(kk''). \tag{3.14}$$

The contribution of all diagrams of the types shown in Figs. 9(b) and 9(c) is given by

$$-\delta_{kk'}\{\sum_{k''}[W_{q\omega}^{(7)}(kk'') + W_{q\omega}^{(8)}(kk'')]+\Lambda_{q\omega}(k)\}. \tag{3.15}$$

We note that the term $-\delta_{kk'}\Lambda_{q\omega}(k)$ in (3.15), plus the sum of all contributions of diagrams for $W_{q\omega}^{[2]}(kk')$ not yet considered, can be written quite simply as

$$-\delta_{kk'}\sum_{k''} W_{q\omega}^{(a)}(kk''). \tag{3.16}$$

Thus, collecting these results and using (3.13), we have

$$W_{q\omega}^{[2A]}(kk') = -\delta_{kk'}\sum_{k''} n_i^2 w_{q\omega}^{(2)}(kk''). \tag{3.17}$$

Finally, the collision term of second order in n_i of the transport equation (1.9) can be written as

$$\sum_{k'} W_{q\omega}^{[2]}(kk')f_{q\omega}(k') = \sum_{k'} n_i^2 w_{q\omega}^{(2)}(kk') \times[f_{q\omega}(k')-f_{q\omega}(k)], \tag{3.18}$$

where $w_{q\omega}^{(2)}(kk')$ is given by (3.13).

IV. STATIC, UNIFORM LIMIT—COMPARISON WITH RESULTS OF LUTTINGER AND KOHN

In this section, we shall discuss the coefficients of the transport equation that we have derived in the previous two sections in the limit $\mathbf{q}\to 0$, then $\omega\to 0$. We remark that we take the $\mathbf{q}\to 0$ limit before the $\omega\to 0$ limit, because this corresponds to the physical situation of a uniform time-independent, but adiabatically switched external field. We also note that in taking the limit $\omega\to 0$, we really mean to take $\mathrm{Re}\omega\to 0$; the $\mathrm{Im}\omega\to 0$ limit is always taken at the very end of the calculation. For convenience, in the rest of this section we shall denote $\mathrm{Im}\omega$ by 2α.

Let us consider the collision function. First note that for real z,

$$\lim_{\omega,q\to 0} G_{qk}(z,\omega) = -2i\alpha|g_k(z+i\alpha)|^2 \tag{4.1}$$

and

$$\lim_{\omega,q\to 0} T_{kk',q}(z,\omega) = |t_{kk'}(z+i\alpha)|^2. \tag{4.2}$$

Then from (2.13) we have

$$\lim_{\omega,q\to 0} w_{q\omega}^{(1)}(kk') = \frac{2\alpha^2}{\pi}\int_{-\infty}^{\infty} dz|g_k(z+i\alpha)|^2 \times|g_{k'}(z-i\alpha)|^2|t_{kk'}(z+i\alpha)|^2. \tag{4.3}$$

Comparing with Eq. (91) of Ref. 2, we see that

$$\lim_{\omega,q\to 0} n_i w_{q\omega}^{(1)}(kk') = \frac{1}{i}J_{kk'}^{(1)} = n_i w_{kk'}, \tag{4.4}$$

where $J_{kk'}^{(1)}$ and $w_{kk'}$ are defined in Ref. 2. Referring to Eqs. (3.5)–(3.8) and to (108)–(111) of Ref. 2, we find

$$\lim_{\omega,q\to 0} W_{q\omega}^{(6)}(kk') = n_i^2[I_{kk'}^{(1)}+I_{kk'}^{(1)*}] - \frac{1}{2\alpha}\sum_{k''} J_{kk''}^{(1)}J_{kk'}^{(1)}, \tag{4.5}$$

$$\lim_{\omega,q\to 0} W_{q\omega}^{(7)}(kk') = n_i^2[I_{kk'}^{(2)}+I_{kk'}^{(2)*}] - \frac{1}{2\alpha}\sum_{k''} J_{kk'}^{(1)}J_{k'k''}^{(1)}, \tag{4.6}$$

$$\lim_{\omega,q\to 0} W_{q\omega}^{(8)}(kk') = n_i^2 I_{kk'}^{(3)} + \frac{1}{2\alpha}\sum_{k''} J_{kk''}^{(1)}J_{k''k'}^{(1)}, \tag{4.7}$$

and

$$\lim_{\omega,q\to 0} W_{q\omega}^{(a)}(kk') = n_i^2 I_{kk'}^{(0)}. \tag{4.8}$$

We thus find from these equations and (3.13) that

$$\lim_{\omega,q\to 0} n_i^2 w_{q\omega}^{(2)}(kk') = \frac{1}{i}K_{kk'}, \tag{4.9}$$

where $K_{kk'}$ is as defined by Eq. (133) of Ref. 2. Thus, up to order n_i^2 our collision function in the limit $\omega,q\to 0$ becomes identical to that of Luttinger and Kohn.[2]

Let us now consider the driving function. Since in the limit $q\rightarrow 0$, there is no difference between transverse and longitudinal fields, we can use (2.32) and find

$$\lim_{q\to 0}\Delta_q(kk')\equiv\Delta(kk')=F_{00}{}^{\alpha}\left(\frac{\partial}{\partial k_\alpha}+\frac{\partial}{\partial k_{\alpha'}}\right)\int_C\frac{dz}{2\pi i}F(z)g_k(z)\times t_{kk'}(z)g_{k'}(z). \quad (4.10)$$

If in Ref. 2 we change from Boltzmann to Fermi statistics, i.e., change their $\rho^0(z)$ to $F(z)$ (we shall say more about this when we discuss the zero-order term), we find that

$$\Delta(kk')=-F_{00}{}^{\alpha}P_{kk'}{}^{\alpha}, \quad (4.11)$$

where $P_{kk'}{}^{\alpha}$ is defined in their Eqs. (152) and (153). From Eq. (2.33) we then see that

$$\lim_{\omega,q\to 0}D_{q\omega}{}^{[1]}(k)=-n_iF_{00}{}^{\alpha}\biggl(P_{kk}{}^{\alpha}-\frac{\alpha}{\pi}\int_{-\infty}^{\infty}dz|g_k(z+i\alpha)|^2\times\{\sum_{k'}[t_{kk'}(z+i\alpha)g_{k'}(z+i\alpha)P_{k'k}{}^{\alpha}+P_{kk'}{}^{\alpha}g_{k'}(z-i\alpha)t_{k'k}(z-i\alpha)]+\sum_{k'k''}t_{kk'}(z+i\alpha)g_{k'}(z+i\alpha)\times P_{k'k''}{}^{\alpha}g_{k''}(z-i\alpha)t_{k''k}(z-i\alpha)\}\biggr). \quad (4.12)$$

Using Eq. (151) of Ref. 2, which defines the quantity $B_k{}^{(1)}$, we have

$$\lim_{\omega,q\to 0}D_{q\omega}{}^{[1]}(k)=-n_iF_{00}{}^{\alpha}P_{kk}{}^{\alpha}+(1/i)B_k{}^{(1)}. \quad (4.13)$$

Finally, let us discuss the zeroth-order term $D_{q\omega}{}^{[0]}(k)$ in the limit $q\rightarrow 0$, $\omega\rightarrow 0$. We have simply

$$\lim_{\omega,q\to 0}D_{q\omega}{}^{[0]}(k)=-F_{00}{}^{\alpha}\frac{\partial}{\partial k_\alpha}F(\epsilon_k)=F_{00}{}^{\alpha}\frac{\partial}{\partial k_\alpha}\int_C\frac{dz}{2\pi i}F(z)g_k(z). \quad (4.14)$$

Notice now that

$$-F_{00}{}^{\alpha}\frac{\partial}{\partial k_\alpha}F(\epsilon_k)-n_iF_{00}{}^{\alpha}P_{kk}{}^{\alpha}\neq i(C_k{}^{(0)}+C_k{}^{(1)}), \quad (4.15)$$

where $C_k{}^{(0)}$ and $C_k{}^{(1)}$ are as found in Appendix B of Ref. 2. If equality did hold in (4.15), then in the limit $\omega,q\rightarrow 0$ our transport equation would be the same as that of Luttinger and Kohn. What is the nature of the difference? To answer this question we note that the above-mentioned authors have effectively used a distribution that corresponds to the same number of electrons whether or not the impurities are present; furthermore, they have used a Boltzmann distribution function. In our work, we have used a constant chemical potential and have also used Fermi statistics. Let us see what happens if we write $D^{[0]}$ in terms of the chemical potential μ_0, which makes the number of electrons in the pure system equal to the number in the impure system with chemical potential μ. To first order in n_i, we have

$$N=-\sum_k\int_C\frac{dz}{2\pi i}\left(F(z)+(\mu_0-\mu)\frac{\partial}{\partial\mu}F(z)\right)g_k(z)=-\sum_k\int_C\frac{dz}{2\pi i}F(z)\{g_k(z)+n_i[g_k(z)]^2t_{kk}(z)\}, \quad (4.16)$$

hence

$$(\mu-\mu_0)=-n_i\sum_k\int_C\frac{dz}{2\pi i}F(z)[g_k(z)]^2t_{kk}(z)\bigg/\sum_k\int_C\frac{dz}{2\pi i}\frac{\partial}{\partial\mu}F(z)g_k(z). \quad (4.17)$$

We then can write

$$\lim_{\omega,q\to 0}D_{q\omega}{}^{[0]}(k)=-F_{00}{}^{\alpha}\frac{\partial}{\partial k_\alpha}\times\left[F_0(\epsilon_k)-(\mu-\mu_0)\int_C\frac{dz}{2\pi i}\frac{\partial}{\partial\mu}F(z)g_k(z)\right], \quad (4.18)$$

where

$$F_0(z)=(e^{\beta(z-\mu_0)}+1)^{-1}. \quad (4.19)$$

Now let us note that if we replace $F(z)$ by its Boltzmann limit

$$F^B(z)=e^{-\beta(z-\mu)}, \quad (4.20)$$

in that limit we then obtain from (4.17) and (4.18),

$$D_{q\omega}{}^{[0]}(k)=-F_{00}{}^{\alpha}\frac{\partial}{\partial k_\alpha}[F_0{}^B(\epsilon_k)(1+n_i\delta)]. \quad (4.21)$$

Here δ is as defined in Eq. (B23) in Ref. 2. (Our expression for δ looks slightly different in *form* from theirs; this is because we have normalized our distribution to N whereas they normalize to unity.) With expression (4.21), we can establish equality in (4.15). Thus we conclude that in the Boltzmann limit and for $\omega,q\rightarrow 0$ our transport equation is identical with the one of Ref. 2 (except for a trivial over-all factor of i). For general Fermi statistics, we get the equation

$$\sum_{k'}(J_{kk'}{}^{(1)}+K_{kk'})[f_{00}(k')-f_{00}(k)]=iF_{00}{}^{\alpha}\left(\frac{\partial}{\partial k_\alpha}F(\epsilon_k)+n_iP_{kk}{}^{\alpha}\right)-B_k{}^{(1)}, \quad (4.22)$$

where $J_{kk'}^{(1)}$, $K_{kk'}$, $P_{kk'}^{\alpha}$, and $B_k^{(1)}$ are as in Ref. 2, except that we replace $\rho^B(z)=e^{-\beta z}/K_0$ occurring there by $F(z)$.

To conclude, we note that our calculation is strictly a result of perturbation theory. However, we have identified certain sums as the iterations of integral equations that functions such as the t matrix satisfy. If we interpret the t matrices, etc., in our expressions as their exact values (rather than the values obtained from the perturbation series for them), we expect our results to be valid even when perturbation theory fails. The fact that our results in the limit $\omega,q\rightarrow 0$ are identical to those of Luttinger and Kohn,[2] which were not based on perturbation theory, strengthens this expectation. In a later paper, we shall obtain the same results we have found here by a nonperturbative method, thus completely confirming this expectation.

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APPENDIX

The quantity P in equation (2.16) is of the general form

$$P(y_1\cdots y_n,x_1\cdots x_n,\omega)$$

$$\equiv\sum_{\nu_1=1}^{n}\sum_{\nu_2=\nu_1}^{n}\cdots\sum_{\nu_{l-1}=\nu_{l-2}}^{n}\prod_{s=1}^{l}\prod_{j=\nu_{s-1}}^{\nu_s}(y_j-x_s-\omega)^{-1},\quad\text{(A1)}$$

with $\nu_0\equiv 1$ and $\nu_l\equiv n$. Here y_j and x_s are real numbers, while ω is complex with a positive imaginary part. Each term in the sum is a product of $n+l-1$ distinct factors $(y_j-x_s-\omega)^{-1}$ with the following restrictions: (i) In each product, every y_j and every x_s must occur in at least one of the factors $(y_j-x_s-\omega)^{-1}$. (ii) If x_s and y_j occur together in a factor $(y_j-x_s-\omega)^{-1}$ of a product, and $x_{s'}$ and $y_{j'}$ occur together in another factor of that product, and if $j>j'$, then $s\geqslant s'$; alternatively, we can say that if $s>s'$, then $j\geqslant j'$. The sum in (A1) is over all possible distinct products conforming to the two restrictions. Note that it follows from the alternate way of stating restriction (ii) that we also have

$$P(y_1\cdots y_n,x_1\cdots x_l,\omega)$$

$$=\sum_{\nu_1=1}^{l}\sum_{\nu_2=\nu_1}^{l}\cdots\sum_{\nu_{n-1}=\nu_{n-2}}^{l}\prod_{j=1}^{n}\prod_{s=\nu_{j-1}}^{\nu_j}(y_j-x_s-\omega)^{-1},\quad\text{(A2)}$$

with $\nu_0\equiv 1$ and $\nu_n\equiv l$.

We shall now prove the lemma

$$P(y_1\cdots y_n,x_1\cdots x_l,\omega)$$

$$=(-1)^n\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\prod_{j=1}^{n}(z+\tfrac{1}{2}\omega-y_j)^{-1}\times\prod_{s=1}^{l}(z-\tfrac{1}{2}\omega-x_s)^{-1}.\quad\text{(A3)}$$

Note that in fact P does not really depend on n in the way the first factor in (A3) might suggest; this superficial sign dependence on n can be eliminated by simply changing the sign of the first n factors within the integral. In (A3) note that the contour can be closed in the upper half-plane (where the poles of the terms involving the x's lie), or in the lower half-plane (where the poles of the terms with the y's are).

We shall prove lemma (A3) by induction. For $l=1$ and arbitrary n, by closing the contour in the upper half-plane, we get from the right-hand side of (A3)

$$\prod_{j=1}^{n}(y_j-x_1-\omega)^{-1},\quad\text{(A4)}$$

which is easily seen to be the same as (A1) for $l=1$. Now for $l=L$ and arbitrary n, the product of the last two factors of (A3) can be written in terms of their difference (for the moment we take $x_L\neq x_{L-1}$), and thus

$$(x_L-x_{L-1})^{-1}(-1)^n\int_{-\infty}^{\infty}\frac{dz}{2\pi i}\times\prod_{j=1}^{n}(z+\tfrac{1}{2}\omega-y_j)^{-1}\prod_{s=1}^{L-2}(z-\tfrac{1}{2}\omega-x_s)^{-1}\times[(z-\tfrac{1}{2}\omega-x_L)^{-1}-(z-\tfrac{1}{2}\omega-x_{L-1})^{-1}].\quad\text{(A5)}$$

Assuming that the lemma (A3) is true for $l=L-1$ (and arbitrary n), we note that (A5) can be written as

$$(x_L-x_{L-1})^{-1}\sum_{\nu_1=1}^{n}\cdots\sum_{\nu_{L-2}=\nu_{L-3}}^{n}\prod_{s=1}^{L-2}\prod_{j=\nu_{s-1}}^{\nu_s}(y_j-x_s-\omega)^{-1}\times\Big[\prod_{j=\nu_{L-2}}^{n}(y_j-x_L-\omega)^{-1}-\prod_{j=\nu_{L-2}}^{n}(y_j-x_{L-1}-\omega)^{-1}\Big].\quad\text{(A6)}$$

The difference of the products in (A6) now can be written, according to (2.9), as

$$\sum_{i=\nu_{L-2}}^{n}\prod_{j=\nu_{L-2}}^{i}(y_j-x_{L-1}-\omega)^{-1}\times\prod_{j=i}^{n}(y_j-x_L-\omega)^{-1}(x_L-x_{L-1}).\quad\text{(A7)}$$

Using (A7) (with the dummy index i changed to ν_{L-1}) in (A6), we obtain (A1) for the case $l=L$. This proves the lemma by induction. Note that we have considered only the case $x_{L-1}\neq x_L$. For the case $x_L=x_{L-1}$ the lemma (A3) still holds, since each of the expressions (A1) and (A3) for $x_{L-1}=x_L$ is equal to its limit as $x_L\rightarrow x_{L-1}$.

The second lemma that we wish to prove is a general form of (2.26). Defining the quantity

$$Q(x_0x_1\cdots x_n)$$
$$\equiv[-F(x_n)][1-F(x_0)]\int_0^\beta d\gamma_1\cdots\int_0^\beta d\gamma_n$$
$$\times\prod_{i=1}^n e^{\gamma_i(x_i-x_{i-1})}\prod_{j=2}^n\{\theta(\gamma_j-\gamma_{j-1})[1-F(x_{j-1})]$$
$$-\theta(\gamma_{j-1}-\gamma_j)F(x_{j-1})\},\quad\text{(A8)}$$

where x_i are real, we shall prove that

$$Q(x_0x_1\cdots x_n)=(-1)^n\int_C\frac{dz}{2\pi i}F(z)\prod_{i=0}^n(z-x_i)^{-1},\quad\text{(A9)}$$

where the contour C is given in Fig. 3.

Employing the method of induction, we first note that for $n=1$, both members of (A9) give for $x_1\neq x_0$

$$[F(x_1)-F(x_0)]/(x_1-x_0).\quad\text{(A10)}$$

We again note that the product of the last two factors in the right-hand side of (A9) can be written in terms of their difference, and thus the right-hand side of (A9) for $n=N$ can be written as

$$(x_{N-1}-x_N)^{-1}(-1)^{n-1}\int_C\frac{dz}{2\pi i}F(z)\prod_{i=0}^{N-2}(z-x_i)^{-1}$$
$$\times[(z-x_N)^{-1}-(z-x_{N-1})^{-1}].\quad\text{(A11)}$$

Assuming (A9) for $n=N-1$, we can write (A11) as

$$[1-F(x_0)]\int_0^\beta d\gamma_1\cdots\int_0^\beta d\gamma_{N-1}\prod_{i=1}^{N-2}e^{\gamma_i(x_i-x_{i-1})}$$
$$\times\prod_{j=2}^{N-1}\{\theta(\gamma_j-\gamma_{j-1})[1-F(x_{j-1})]-\theta(\gamma_{j-1}-\gamma_j)F(x_{j-1})\}$$
$$\times\{(x_N-x_{N-1})^{-1}[e^{\gamma_{N-1}(x_N-x_{N-2})}F(x_N)$$
$$-e^{\gamma_{N-1}(x_{N-1}-x_{N-2})}F(x_{N-1})]\}.\quad\text{(A12)}$$

Now it is easily checked that the last factor in the second curly bracket in (A12) can be rewritten

$$[-F(x_N)]e^{\gamma_{N-1}(x_{N-1}-x_{N-2})}\int_0^\beta d\gamma_N e^{\gamma_N(x_N-x_{N-1})}$$
$$\times\{\theta(\gamma_N-\gamma_{N-1})[1-F(x_{N-1})]$$
$$-\theta(\gamma_{N-1}-\gamma_N)F(x_{N-1})\}.\quad\text{(A13)}$$

Thus, inserting (A13) into (A12) and comparing it with (A8), we see that (A9) is proven for $n=N$, if it is valid for $n=N-1$. Equation (A9) is thus proven by induction. Note that we have (implicitly) considered only the case $x_{N-1}\neq x_N$. For the case $x_{N-1}=x_N$ the lemma (A9) still holds, since both expressions (A8) and (A9) for $x_{N-1}=x_N$ are equal to their limits as $x_N\rightarrow x_{N-1}$.